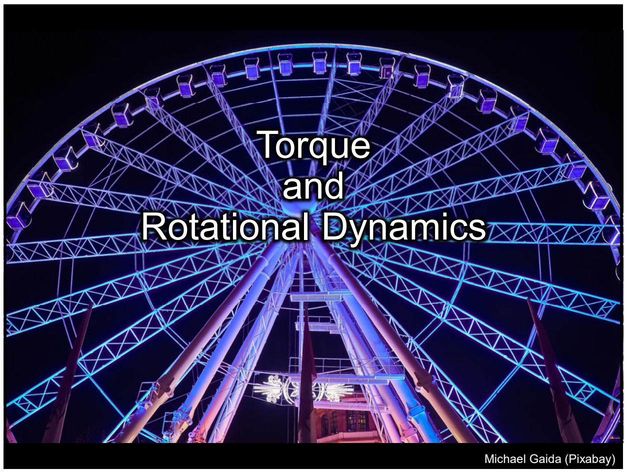


Torque and Rotational Dynamics

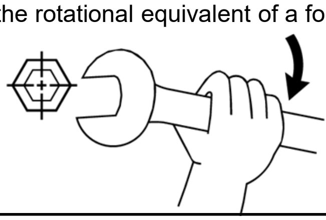
Michael Gaida (Pixabay)

Torque

Alexlmx (Adobe Stock)

Torque

- Torque is a measure of the effectiveness of a force in changing or accelerating a rotation (changing the angular velocity over a period of time).
  - It is the rotational equivalent of a force.



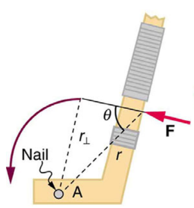
Jeff Stein (Pixabay)

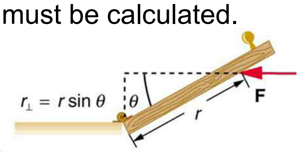
- The rotation is around a pivot point.
- The distance from the pivot point to the point where the force acts is called the lever arm or moment arm.
- The applied force must be perpendicular to the lever arm to cause rotation.

- The magnitude of torque is defined to be

$$\tau = r_{\perp} F$$

- If the force is not perpendicular to the lever arm, then the perpendicular component must be calculated.

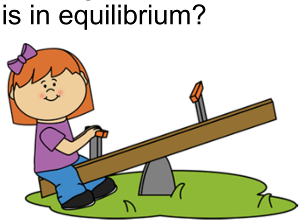


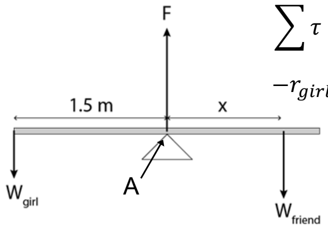


$$\tau = rF\sin\theta$$

Example

A 50.0 kg girl is sitting at the end of a 3.0 m long see-saw. How far from the center must her 60.0 kg friend sit such that the see-saw is in equilibrium?





$$\sum \tau = 0 \quad \text{about A}$$

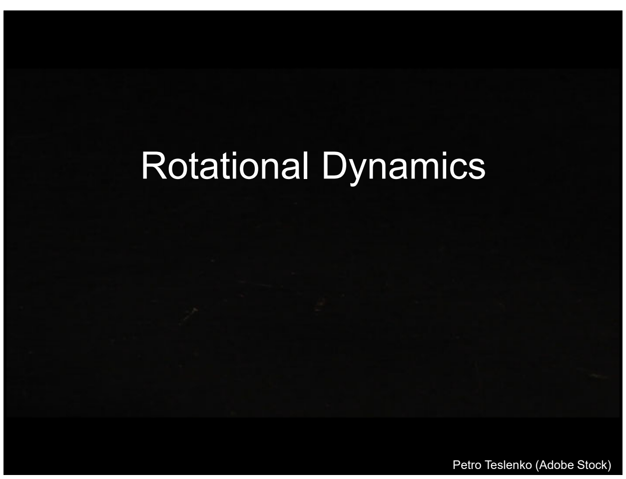
$$-r_{girl}F_{girl} + r_{friend}F_{friend} = 0$$

$$r_{friend} = \frac{r_{girl}m_{girl}g}{m_{friend}g}$$

$$r_{friend} = \frac{(1.5)(50)}{(60)} = 1.3 \text{ m}$$

Equilibrium

- An object is in equilibrium if it is stationary or moving with a constant velocity.
  - Translational equilibrium
    - Net external force on the system must be zero.
  - Rotational equilibrium
    - Net external torque on the system must be zero.



Rotation Angle

- When an object rotates, all the points along the radius move through the same angle in the same amount of time.
- Therefore, it is convenient to measure position, velocity, and acceleration in terms of angle.

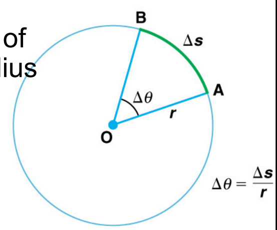
Ferris wheel animated icon created by Freepik - Flaticon

OpenStax, Rice University (CC BY 4.0)

Angular Rotation

- We define the rotation angle $\Delta\theta$ to be the ratio of the arc length to the radius of curvature.

$$\Delta\theta = \frac{\Delta s}{r}$$



The arc length Δs is the distance traveled along a circular path and r is the radius of curvature of the circular path.

OpenStax, Rice University (CC BY 4.0)

- For one complete revolution, the arc length is the circumference of a circle of radius r.
- The circumference of a circle is $2\pi r$.
- Therefore, for one complete revolution

$$\Delta\theta = \frac{\Delta s}{r} = \frac{2\pi r}{r} = 2\pi$$

- This defines the units we use to measure angular rotation, **radians (rad)**.

$$2\pi \text{ rad} = 1 \text{ revolution} = 360°$$

Angular Velocity

- Rate of change of an angle.

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Units: rad s^{-1}

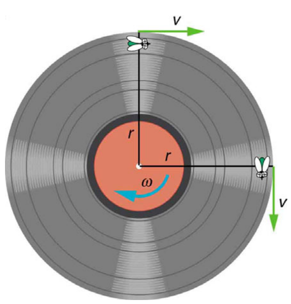
- Angular velocity ω is analogous to linear velocity v.

- A particle moves an arc length of Δs in time Δt.
- The velocity of the object is $v = \frac{\Delta s}{\Delta t}$
- From the definition of angular rotation $\Delta s = r\Delta\theta$
- Substituting gives $v = \frac{r\Delta\theta}{\Delta t} = r\omega$

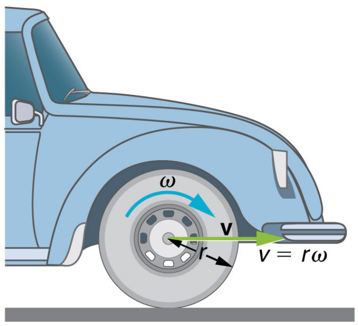
$$v = \omega r \quad \text{or} \quad \omega = \frac{v}{r}$$

- This relationship tells us two things…

- An object moving with an angular velocity ω has a tangential (linear) velocity at any point is equal to ωr.



- A rolling object with a linear velocity of v is rotating with an angular velocity of ω.



Angular Acceleration

- Angular acceleration is defined as the rate of change of angular speed.

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

Units: rad/s^2

- Angular acceleration is related to translational acceleration.

$$a = \alpha r$$

Kinematics of Rotational Motion

- Kinematics for rotational motion is completely analogous to translational kinematics.

Rotational	Translational
$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x = x_0 + v_{x0} t + \frac{1}{2} a_x t^2$
$\omega = \omega_0 + \alpha t$	$v_x = v_{x0} + a_x t$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v_x^2 = v_{x0}^2 + 2a_x(x - x_0)$

Example

A wheel is rotated from rest with an angular acceleration of 8.0 rad/s^2. It accelerates for 5.0 s. Calculate

a) the angular speed.

b) the number of revolutions that the wheel has rotated through.

a) $\omega = \omega_0 + \alpha t$

$\omega = (8)(5) = 40 \text{ rad/s}$

b) $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$

$\theta = \frac{1}{2}(8)(5)^2 = 100 \text{ rad}$

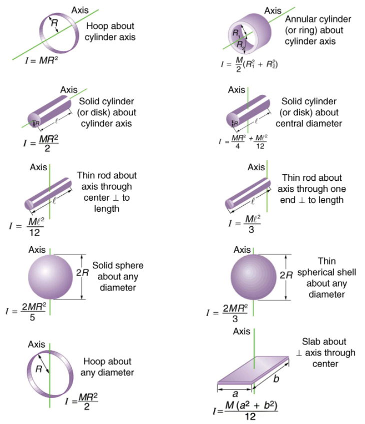
$\text{rotations} = \frac{100}{2\pi} = 16$

Moment of Inertia

- The moment of inertia, I, of an object is defined as the sum of mr^2 for all the point masses of which it is composed.

$$I = \sum mr^2$$

- Moment of inertia is analogous to mass in translational motion.
- The moment of inertia for any object depends on the chosen axis.
- Units: $\text{kg}\cdot\text{m}^2$

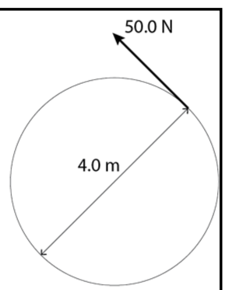


- The general relationship among torque, moment of inertia, and angular acceleration is

$$\vec{\alpha} = \frac{\sum \vec{\tau}}{I} = \frac{\vec{\tau}_{net}}{I}$$

Example

A 45 kg child is sitting on the edge of a 60.0 kg merry-go-round with a diameter of 4.0 m for one rotation. A constant force of 50.0 N is applied tangentially to the edge. The moment of inertia is $\frac{1}{2}mr^2$. Calculate the angular speed of the merry-go-round.



$$\vec{\alpha} = \frac{\vec{\tau}_{net}}{I} \qquad \omega^2 = \omega_0^2 + 2\alpha\theta \qquad \tau = rF\sin\theta$$

$$\alpha = \frac{\omega^2}{2\theta}$$

$$I_m = \frac{1}{2}mr^2 \qquad I_c = mr^2$$

$$I_{total} = \frac{1}{2}m_m r^2 + m_c r^2$$

$$\frac{\omega^2}{2\theta} = \frac{rF}{\frac{1}{2}m_m r^2 + m_c r^2}$$

$$\omega = \sqrt{\frac{2\theta rF}{\frac{1}{2}m_m r^2 + m_c r^2}}$$

$$\omega = \sqrt{\frac{2\theta rF}{\frac{1}{2}m_m r^2 + m_c r^2}}$$

$$\omega = \sqrt{\frac{2(2\pi)(2)(50)}{\frac{1}{2}(60)(2)^2 + (45)(2)^2}} = 1.8 \text{ rad/s}$$